MATH 122B: HOMEWORK 1

Suggested due date: August 8th, 2016

(1) Use the definition to find $f'(z)$ where $f(z) = \frac{1}{z}$, $z \neq 0$.

(2) Define the complex sine and cosine function by

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \text{and } \cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \qquad z \in \mathbb{C}.$$

Verify that $\sin(z)$ and $\cos(z)$ are holomorphic.

(3) Show that there are no holomorphic functions of the form $f = u + iv$ with $u = x^2 + y^2$.

(4) Let $f(z)$ be holomorphic in a domain D. Show that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2.$$

Note: $|z|^2 = z\bar{z}$.

(5) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0. \end{cases}$$

Verify that the Cauchy-Riemann equations for f are satisfied at $z = 0$, but $f'(0)$ does not exist.

(6) If g is holomorphic on $D \subset \mathbb{C}$ and f is holomorphic on $g(D)$, show that $h(z) = f(g(z))$ is holomorphic. What if f is not holomorphic?

(7) **More challenging exercises** Let

$$f(z) = \begin{cases} e^{-\frac{1}{z^4}}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Show that the Cauchy-Riemann equations hold in $\mathbb{C}$ however $f(z)$ is not differentiable in $\mathbb{C}$. Where is f holomorphic? [Hint: Compute the partial derivatives at $z = 0$ by definition]

(8) Let f be a continuous function on the interval $[0, 1]$. From the definition of the derivative, show that the function

$$F(z) = \int_0^1 e^{izt} f(t)dt$$

is holomorphic on $\mathbb{C}$ and compute its derivative. [Hint: The series expansion of $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, with the convention $0! = 1$.]

Solutions

(1) Compute

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{z+h}-\frac{1}{z}\right)=-\frac{1}{z^2}$$

(2) We will verify the Cauchy-Riemann equation for

$$\begin{aligned}2\cos(z)=e^{i(x+iy)}+e^{-i(x+iy)}&=e^{-y}e^{ix}+e^{-ix}e^{y}\\&=e^{-y}(\cos(x)+i\sin(x))+e^{y}(\cos(x)-i\sin(x))\\&=(e^{-y}+e^{y})\cos(x)+i(e^{-y}-e^{y})\sin(x)\\&=u+iv\end{aligned}$$

Then

$$\begin{aligned}u_x&=-\sin(x)(e^{-y}+e^{y})=y_v\\u_y&=\cos(x)(e^{y}-e^{-y})=-v_x.\end{aligned}$$

(3) Use Cauchy Riemann equations, get contradiction.

(4) Show $\frac{1}{4}\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

(5) We need to compute the partial derivatives at 0, i.e.

$$\lim_{h\to 0}\frac{u(h,0)-u(0,0)}{h}$$

etc. To find the relevant components, we know $f(x,0)=u(x,0)+iv(x,0)$, so we need to find the real and imaginary parts. Now

$$f(x,0)=x=x+i0=u(x,0)+iv(x,0)$$

so $u(x,0)=x$ and $v(x,0)=0$. Repeat this for $f(0,y)$ to compute the other two, then compute the partial derivatives by definition. To show $f'(0)$ does not exist, take $h=\varepsilon e^{i\theta}$ with $\varepsilon\to 0$. Then

$$\lim_{\varepsilon\to 0}\frac{f(h)}{h}=e^{-i4\theta}$$

so different θ give different limits.

(6) By chain rule,

$$\frac{\partial h}{\partial \bar{z}}=\frac{\partial f}{\partial g}\frac{\partial g}{\partial \bar{z}}+\frac{\partial f}{\partial \bar{g}}\frac{\partial \bar{g}}{\partial \bar{z}}=0$$

(7) The CR equations hold for $z\neq 0$ since it is a composition of holomorphic functions. At 0, we need to take the partial derivatives by definition. The method is the same as problem 5. It is not differentiable at 0 since it is not even continuous there, for instance consider the path $\varepsilon e^{i\pi/4}$, $\varepsilon\to 0$.

(8) We have

$$\lim_{h\to 0}\left|\frac{1}{h}(F(z+h)-F(z))-\int_0^1 ite^{izt}f(t)dt\right|=|h|e^{|h|}\int_0^1|f(t)e^{izt}|dt\to 0$$